

Deformed Hamilton-Jacobi Method in Covariant Quantum Gravity Effective Models

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Abstract

We first briefly revisit the original Hamilton-Jacobi method and show that the Hamilton-Jacobi equation for the action I of tunnelings of a fermionic particle from a charged black hole can be written in the same form as that of a scalar particle. For the low energy quantum gravity effective models which respect covariance of the curved spacetime, we derive the deformed model-independent KG/Dirac and Hamilton-Jacobi equations using the methods of effective field theory. We then find that, to all orders of the effective theories, the deformed Hamilton-Jacobi equations can be obtained from the original ones by simply replacing the mass of emitted particles m with a parameter m_{eff} that includes all the quantum gravity corrections. Therefore, in this scenario, there will be no corrections to the Hawking temperature of a black hole from the quantum gravity effects if its original Hawking temperature is independent of the mass of emitted particles. As a consequence, our results show that breaking covariance in quantum gravity effective models is a key for a black hole to have the remnant left in the evaporation.

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I. INTRODUCTION

Hawking radiation is a theoretical argument proposed by Stephen Hawking for the existence of thermal radiation emitted by black holes. The standard Hawking formula was first obtained in the frame of quantum field theory in curved spacetime[1]. Afterward, there are various methods for deriving Hawking radiation and calculating Hawking temperature. Among them is a semiclassical method of modeling Hawking radiation as a tunneling effect. This method was first proposed by Kraus and Wilczek[2, 3]. They employed the dynamical geometry approach to calculate the imaginary part of the action for the tunneling process of s-wave emission across the horizon and related it to the Boltzmann factor for the emission at the Hawking temperature. Due to back reaction effects included, this procedure gives a correction to the standard Hawking temperature formula, which speeds up the process of black holes' evaporation. An alternative way to calculate the imaginary part of the action

is the Hamilton-Jacobi method[4, 5]. Neglecting the self-gravitation, this method assumes the action of an emitted particle satisfies the relativistic Hamilton-Jacobi equation. Taking the symmetries of the metric into account, one can adopt an appropriate ansatz for the form of the action. Solving the Hamilton-Jacobi equation turns out to recover the standard Hawking temperatures.

However, the original Hamilton-Jacobi method proposed in [4, 5] is only confined to the semiclassical approximation and carried out in the framework of the classical general relativity. Beyond the original Hamilton-Jacobi method, there could be some corrections. For simplicity, we use the case of emission of massless scalar with energy E from the Schwarzschild black hole with mass M to illustrate these corrections. In this paper, we set $G = c = 1$, where the Planck constant \hbar is of the order of square of the Planck Mass M_p . In these units, the black hole's horizon $r_H = 2M$. The potential corrections to the original Hamilton-Jacobi method are given as follows:

- (a) Back-reaction effects, which gives the correction $\sim \frac{E}{M}$. In the Hamilton-Jacobi method, we assume that the metric is fixed and the field is free. Thus, back-reaction effects are disregarded. However, in Parikh and Wilczek method[2, 3], there are back-reaction effects to ensure energy conservation during the emission of a particle via tunneling through the horizon. These corrections lead to non-thermal corrections to the black-hole radiation spectrum.
- (b) Higher order WKB corrections. Using WKB approximation method, the lowest order of the equation of motion describing a particle moving in the black hole gives the Hamilton-Jacobi equation. The WKB approximation breaks down when the de Broglie wavelength of the particle, $\lambda_p \sim \frac{\hbar}{E}$, becomes comparable to the horizon of black hole, r_H . Therefore, the ratio $\frac{\lambda_p}{r_H} \sim \frac{\hbar}{EM}$ controls the amount of higher order WKB corrections. However, several authors[6–8] argued that the tunneling method yields no higher-order corrections to the Hawking temperature.
- (c) Quantum gravity corrections. The original Hawking formula predicts the complete evaporation of black holes, which leads to the black hole information paradox (for reviews see [9, 10]). Solving the paradox needs the breakdown of the semiclassical description at the Planck scale. A theory of quantum gravity must be used to describe

the final state of black hole evaporation. Although a number of quantum gravity theories are proposed, there is still no complete and consistent quantum theory of gravity. Therefore, in the absence of a full quantum description of the black hole evaporation, one uses effective models to describe the quantum gravitational behavior. For example, various theories of quantum gravity, such as string theory, loop quantum gravity and quantum geometry, predict the existence of a minimal length. An effective model to realize this minimal length is the generalized uncertainty principle (GUP). Adler *et al.*[11] showed that incorporating the GUP into the derivation of Hawking temperature could predict the existence of black hole remnants. Recently, the GUP modified Hamilton-Jacobi equation for fermions in curved spacetime have been introduced and the corrected Hawking temperatures have been derived[12–17]. While in the Parikh-Wilczek tunneling picture, the Planck-scale corrections to the black-hole radiation spectrum, due to the GUP[18], the noncommutative geometry[19, 20], and, a modification of the energy-momentum dispersion relation caused by the quantum gravity effects[21], have also been studied.

The existence of the minimal length implied by the quantum gravity theory could lead to Planck scale departures from Lorentz symmetry. The deformed commutation relations and modified dispersion relations proposed as low-energy quantum gravity effects could deform or break the Lorentz symmetry. Effective models incorporating with such deformed commutation relations or dispersion relations hence are not covariant in spacetime[22–25]. The deformed non-covariant Hamilton-Jacobi equation have been discussed in[12–17]. Although covariance is not required for constructing low-energy quantum gravity effective models, there are indeed some attempts to introduce covariant models[26, 27]. The gauge theories with incorporation of the GUP in [27] are both covariant and gauge invariant. In addition, the Klein-Gordon equation described by Quesne-Tkachuk Lorentz-covariant deformed algebra has been given in [28]. So far, the quantum gravity corrections to the Hamilton-Jacobi method in these covariant effective models have not been discussed. Thus, we will investigate the quantum gravity corrections to the Hamilton-Jacobi method in the covariant low energy quantum gravity effective models in a model-independent way in this paper. To achieve this aim, we will work with an effective theory that is valid below a scale Λ , at which the quantum gravity effects becomes important. As usual, we assume $\Lambda \sim M_p$. We also

assume that the effective theory respects covariance of the curved spacetime and $U(1)$ gauge invariance of the charged black hole. Since the particles tunneling through the horizon are treated as free particles in the original Hamilton-Jacobi method, a natural assumption is that there are no self-interacting effective operators in the effective field theory.

Using the effective theory method, we will show that there are no corrections to the Hawking temperature of a black hole from the quantum gravity effects in the covariant effective models if its original Hawking temperature is independent of the mass of emitted particles. For these purposes, this paper is organized as follows. In Section II, we briefly review the original Hamilton-Jacobi method and show the fermionic Hamilton-Jacobi equation for the action I can be written in the same form of the scalar one. After constructing scalar(fermionic) effective field theories respecting covariance of coordinate spacetime, we derive the deformed model-independent KG/Dirac and Hamilton-Jacobi equations in Section III. Section IV is devoted to our discussion and conclusion.

II. HAMILTON-JACOBI METHOD

In this section, we briefly review how to calculating the imaginary part of the action making use of the Hamilton-Jacobi equation [4]. This semiclassical method models Hawking radiation as a tunneling through the horizon. Using the WKB approximation, the tunneling probability for the classically forbidden trajectory through the horizon is given by:

$$\Gamma \propto \exp\left(\frac{-2\text{Im } I}{\hbar}\right), \quad (1)$$

where I is the classical action of the trajectory. One can relate Γ to the Boltzmann factor for the emission from the black hole to get Hawking temperature.

A. Scalar Field

The equation satisfied by the scalar field is

$$D^\mu D_\mu \phi + \frac{m^2}{\hbar^2} \phi = \left(\nabla^\mu + \frac{ie}{\hbar} A^\mu\right) \left(\nabla_\mu + \frac{ie}{\hbar} A_\mu\right) \phi + \frac{m^2}{\hbar^2} \phi = 0, \quad (2)$$

where ∇_μ is the covariant derivative of the black hole and A_μ is its electromagnetic potential. Making the ansatz for ϕ which is

$$\phi = \exp\left(\frac{iI}{\hbar}\right), \quad (3)$$

and substituting it into eqn. (2), one expands eqn. (2) in powers of \hbar and finds to the lowest order

$$(\partial^\mu I + eA^\mu)(\partial_\mu I + eA_\mu) - m^2 = 0. \quad (4)$$

Eqn. (4) is just the Hamilton-Jacobi equation satisfied by a scalar particle of mass m moving in the black hole with the electromagnetic potential A_μ . The solution to eqn. (4) is the action of the scalar's classically forbidden trajectory through the horizon.

To illustrate how the Hamilton-Jacobi method works, we consider Hawking radiation in the $(1+1)$ dimensional Schwarzschild black hole with line element (with $c = G = 1$)

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2. \quad (5)$$

Thus, eqn. (4) becomes

$$\left(1 - \frac{2M}{r}\right)^{-1} (\partial_t I)^2 - \left(1 - \frac{2M}{r}\right) (\partial_r I)^2 - m^2 = 0, \quad (6)$$

where we use $A_\mu = 0$ for the Schwarzschild black hole. Using the method of separation of variables, we find that the solution to the above equation is

$$I_\pm = -Et \mp \int \frac{dr}{1 - \frac{2M}{r}} \sqrt{E^2 - m^2 \left(1 - \frac{2M}{r}\right)}, \quad (7)$$

where E is a constant and $+$ ($-$) corresponds to the outgoing (ingoing) solutions. Choosing the contour to lie in the upper complex plane, one gets that the imaginary part of I is

$$\text{Im } I_\pm = \mp \int_{2M-\epsilon}^{2M+\epsilon} \frac{dr}{1 - \frac{2M}{r}} \sqrt{E^2 - m^2 \left(1 - \frac{2M}{r}\right)} = \pm 2\pi EM. \quad (8)$$

Thus, the tunneling rate is

$$\Gamma = \frac{P_{(emission)}}{P_{(absorption)}} = \frac{\exp(-2 \text{Im } I_+)}{\exp(-2 \text{Im } I_-)} = \exp(-8\pi ME). \quad (9)$$

Comparing the above equation with the Boltzmann factor at the Hawking temperature near the event horizon gives

$$T = \frac{1}{8\pi M}. \quad (10)$$

B. Fermion Field

In curved spacetime, the Dirac equation for a spin-1/2 fermion with an electromagnetic field A_μ takes on the form as

$$i\gamma_\mu \left(\partial^\mu + \Omega^\mu + \frac{ie}{\hbar} A_\mu \right) \psi - \frac{m}{\hbar} \psi = 0, \quad (11)$$

where $\Omega_\mu \equiv \frac{i}{2} \omega_\mu^{ab} \Sigma_{ab}$, Σ_{ab} is the Lorentz spinor generator, ω_μ^{ab} is the spin connection and $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$. The Greek indices are raised and lowered by the curved metric $g_{\mu\nu}$, while the Latin indices are governed by the flat metric η_{ab} . To obtain the Hamilton-Jacobi equation for a fermion, the ansatz for ψ is assumed as

$$\psi = \exp \left(\frac{iI}{\hbar} \right) v, \quad (12)$$

where v is a slowly varying spinor amplitude. Substituting eqn. (12) into eqn. (11), we find to the lowest order of \hbar

$$\gamma_\mu (\partial^\mu I + eA^\mu) v = -mv, \quad (13)$$

which is the Hamilton-Jacobi equation satisfied by a fermion particle of mass m moving in the black hole with the electromagnetic potential A_μ . Solving eqn. (13) gives us the classical action I . Multiplying both sides of eqn. (13) from the left by $\gamma_\nu (\partial^\nu I + eA^\nu)$ and then using eqn. (13) to simplify the RHS, one gets

$$\gamma_\nu (\partial^\nu I + eA^\nu) \gamma_\mu (\partial^\mu I + eA^\mu) v = m^2 v. \quad (14)$$

Manipulating the LHS of the above equation by using $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$, we have

$$[(\partial^\mu I + eA^\mu) (\partial_\mu I + eA_\mu) - m^2] v = 0. \quad (15)$$

Since v is nonzero, the Hamilton-Jacobi equation satisfied by the classical action I for a fermion finally becomes

$$(\partial^\mu I + eA^\mu) (\partial_\mu I + eA_\mu) - m^2 = 0, \quad (16)$$

which is the same as the Hamilton-Jacobi equation for a scalar with the same mass m , namely eqn. (4).

III. HAMILTON-JACOBI EQUATION WITH INCORPORATION OF QUANTUM GRAVITY EFFECTS

To incorporate the quantum effects into the original Hamilton-Jacobi Method, we need to derive the KG/Dirac equations in low energy quantum gravity effective models. In this section, we will calculate the deformed KG/Dirac and Hamilton-Jacobi Equation with incorporating quantum gravity effects in a covariant way by making using of effective theory Method.

A. Model-Independent Covariant Deformed KG/Dirac Equations

To set notation, the effective Lagrangian involving the scalar field ϕ (fermion field ψ) in the background of a $(D + 1)$ -dimensional black hole with the electromagnetic potential A_μ is given by

$$\mathcal{L}_{eff}^{s(f)} = \sum_{n,j} \frac{C_{n,j}^{s(f)}}{\Lambda^{n-(D+1)}} \mathcal{O}_{n,j}^{s(f)}, \quad (17)$$

where $s(f)$ denotes the scalar (fermion), the $n \geq D + 1$ denotes the operator dimension, j runs over all independent operators of a given dimension. The lowest dimensional operator with $n = D + 1$ is the original free field Lagrangian in curved spacetime with the electromagnetic potential A_μ

$$\mathcal{O}_{D+1}^s = -\phi^+ \left(D^\mu D_\mu + \frac{m^2}{\hbar^2} \right) \phi, \quad (18)$$

$$\mathcal{O}_{D+1}^f = \bar{\psi} \left[i D_\mu^f \gamma^\mu - \frac{m}{\hbar} \right] \psi, \quad (19)$$

where $D_\mu \equiv \nabla^\mu + \frac{ie}{\hbar} A^\mu$, $D_\mu^f \equiv \partial_\mu + \Omega_\mu + \frac{ie}{\hbar} A_\mu$, and m is the mass of the particle. For the fermion field ψ , the basis of independent effective operators with $n = D + 2$ is given by

$$\begin{aligned} \mathcal{O}_{D+2,1}^f &= \hbar \bar{\psi} (\gamma^\nu D_\nu^f) (\gamma^\mu D_\mu^f) \psi, \quad \mathcal{O}_{D+2,2}^f = \hbar \bar{\psi} D^{f,\mu} D_\mu^f \psi, \\ \mathcal{O}_{D+2,3}^f &= i m \bar{\psi} (\gamma^\mu D_\mu^f) \psi, \quad \mathcal{O}_{D+2,4}^f = \frac{m^2}{\hbar} \bar{\psi} \psi. \end{aligned} \quad (20)$$

For the scalar field ϕ , the operator with $n = D + 2$ which are gauge-invariant and covariant is

$$\mathcal{O}_{D+2}^s = m \phi^+ (D^\mu D_\mu) \phi. \quad (21)$$

If we truncate the scalar effective theory at $\mathcal{O}\left(\frac{1}{\Lambda}\right)$, by redefining the scalar field ϕ , it is easy to see that the truncated effective theory is equivalent to the original field theory with redefined mass. So we need effective operators at $\mathcal{O}\left(\frac{1}{\Lambda^2}\right)$ to produce nontrivial results. The basis of independent effective operators with $n = D + 3$ is

$$\begin{aligned}\mathcal{O}_{D+3,1}^s &= \hbar^2 \phi^+ (D^\mu D_\mu D^\nu D_\nu) \phi, \quad \mathcal{O}_{D+3,2}^s = \hbar^2 \phi^+ (D^\mu D^\nu D_\nu D_\mu) \phi, \\ \mathcal{O}_{D+3,3}^s &= \hbar^2 \phi^+ (D^\mu D^\nu D_\mu D_\nu) \phi, \quad \mathcal{O}_{D+3,4}^s = m^2 \phi^+ (D^\mu D_\mu) \phi, \quad \mathcal{O}_{D+3,5}^s = \frac{m^4}{\hbar^2} \phi^+ \phi.\end{aligned}\quad (22)$$

Define the action $S_{eff}^{(f)} = \int d^{D+1}x \sqrt{|g|} \mathcal{L}_{eff}^{(f)}$. Varying $S_{eff}^{(f)}$ with respect to $\phi^+ (\bar{\psi})$ gives the deformed KG(Dirac) equation of $\phi (\psi)$. Thus, the deformed KG equation to $\mathcal{O}\left(\frac{1}{\Lambda^2}\right)$ is

$$\begin{aligned}& -D^\mu D_\mu \phi - \frac{m^2}{\hbar^2} \phi + C_{D+2}^s \frac{m}{\Lambda} D^\mu D_\mu \phi \\ & + \frac{C_{D+3,1}^s \hbar^2}{\Lambda^2} D^\mu D_\mu D^\nu D_\nu \phi + \frac{C_{D+3,2}^s \hbar^2}{\Lambda^2} D^\mu D^\nu D_\nu D_\mu \phi \\ & + \frac{C_{D+3,3}^s \hbar^2}{\Lambda^2} D^\mu D^\nu D_\mu D_\nu \phi + \frac{C_{D+3,4}^s m^2}{\Lambda^2} D^\mu D_\mu \phi + \frac{C_{D+3,5}^s m^4}{\Lambda^2 \hbar^2} \phi = 0,\end{aligned}\quad (23)$$

and the deformed Dirac equation to $\mathcal{O}\left(\frac{1}{\Lambda}\right)$ is

$$\begin{aligned}& i\gamma^\mu D_\mu^f \psi - \frac{m}{\hbar} \psi + \frac{C_{D+2,1}^f \hbar}{\Lambda} (\gamma^\nu D_\nu^f) (\gamma^\mu D_\mu^f) \psi \\ & + \frac{C_{D+2,2}^f \hbar}{\Lambda} D^{f,\mu} D_\mu^f \psi + \frac{iC_{D+2,3}^f m}{\Lambda} (\gamma^\mu D_\mu^f) \psi + \frac{C_{D+2,4}^f m^2}{\Lambda \hbar} \psi = 0.\end{aligned}\quad (24)$$

B. Deformed Scalar Hamilton-Jacobi Equation

To find the classical action I by using WKB approximation to solve eqn. (23), we make the ansatz for ϕ as before

$$\phi = \exp\left(\frac{iI}{\hbar}\right). \quad (25)$$

Substituting it into eqn. (23), one expands eqn. (23) in powers of \hbar and finds to the lowest order

$$A(\partial^\mu I + eA^\mu)(\partial_\mu I + eA_\mu) - Bm^2 + \frac{C}{\Lambda^2} [(\partial^\mu I + eA^\mu)(\partial_\mu I + eA_\mu)]^2 = 0, \quad (26)$$

where $A = 1 - \frac{C_{D+2}^s m}{\Lambda} - \frac{C_{D+3,4}^s m^2}{\Lambda^2}$, $B = 1 - \frac{C_{D+3,5}^s m^2}{\Lambda^2}$, and $C \equiv C_{D+3,1}^s + C_{D+3,2}^s + C_{D+3,3}^s$.

Solving eqn. (26) gives

$$(\partial^\mu I + eA^\mu)(\partial_\mu I + eA_\mu) = m_{eff,\pm}^2, \quad (27)$$

where

$$m_{eff,\pm}^2 \equiv \frac{-A \pm \sqrt{A^2 + 4BCm^2\Lambda^{-2}}}{2C}\Lambda^2. \quad (28)$$

When $m/\Lambda \rightarrow 0$, we find $m_{eff,+}^2 \sim m^2$ and $m_{eff,-}^2 \sim -\frac{2}{C}\Lambda^2$. As $\Lambda \gg m$, eqn. (27) gives $\partial^\mu I \sim \Lambda$ for $m_{eff,-}^2$ and hence the action I highly oscillates in spacetime. One may argue such action is not physical and hence could be discarded by using the low-momentum consistency condition proposed in [29]. Alternatively, one wants to recover the original results when $\Lambda \rightarrow \infty$. However, as $\Lambda \rightarrow \infty$, the solution to in eqn. (27) with $m_{eff,-}^2$ blows up while eqn. (27) with $m_{eff,+}^2$ becomes eqn. (4) since $m_{eff,+}^2 \rightarrow m^2$. Therefore, we pick $m_{eff,+}^2$ in eqn. (27). Comparing eqn. (27) with eqn. (4), we find that all the quantum gravity contributions to the deformed scalar Hamilton-Jacobi equation are included in one effective parameter, m_{eff}^2 .

C. Deformed Fermionic Hamilton-Jacobi Equation

To obtain the Hamilton-Jacobi equation for the classical action, the ansatz for ψ takes the form of

$$\psi = \exp\left(\frac{iI}{\hbar}\right) v, \quad (29)$$

where v is a vector function of the spacetime. Substituting eqn. (29) into eqn. (24), we find to the lowest order of \hbar

$$-A\gamma_\mu (\partial^\mu I + eA^\mu) v = \left[Bm + \frac{C}{\Lambda} (\partial^\mu I + eA^\mu) (\partial_\mu I + eA_\mu) \right] v, \quad (30)$$

where $A = 1 + \frac{C_{D+2,3}^f m}{\Lambda}$, $B = 1 - \frac{C_{D+2,4}^f m}{\Lambda}$, and $C \equiv C_{D+2,1}^f + C_{D+2,2}^f$. Multiplying both sides of eqn. (30) from the left by $-A\gamma_\nu (\partial^\nu I + eA^\nu)$ and using eqn. (30) on RHS, we obtain

$$A^2 (\partial^\mu I + eA^\mu) (\partial_\mu I + eA_\mu) = \left[Bm + \frac{C}{\Lambda} (\partial^\mu I + eA^\mu) (\partial_\mu I + eA_\mu) \right]^2. \quad (31)$$

Solving eqn. (31), we also find

$$(\partial^\mu I + eA^\mu) (\partial_\mu I + eA_\mu) = m_{eff}^2, \quad (32)$$

where m_{eff}^2 is a function of m , Λ , A , B , and C and is chosen as $m_{eff}^2 \rightarrow m^2$ as $\Lambda \rightarrow \infty$. Similarly, all the quantum gravity contributions to the deformed fermionic Hamilton-Jacobi equation are included in one effective parameter, m_{eff}^2 .

D. Deformed Hamilton-Jacobi Equation to All Orders

We have shown that the deformed scalar(fermionic) Hamilton-Jacobi Equation can be written as the form of eqn. (32) up to the order of $\frac{1}{\Lambda^2}$ ($\frac{1}{\Lambda}$). Here, we will show that the deformed scalar(fermionic) Hamilton-Jacobi Equation can also be written as the form of eqn. (32) to all orders of the effective theories.

For a scalar field, the effective operator $\mathcal{O}_{n,j}^s$ must contain even number of D_μ to be covariant. Since $\mathcal{O}_{n,j}^s$ contains two ϕ , we find

$$\mathcal{O}_{n,j}^s = \hbar^{2q-2} m^p \phi^+ \mathcal{C} (D_{\mu_1} \cdots D_{\mu_{2q}}) \phi, \quad (33)$$

where integers $p, q \geq 0$, $2q + p = n - D + 1$, $j = \{\mathcal{C}, p\}$ and \mathcal{C} denotes any possible way of contracting $\mu_1 \cdots \mu_{2q}$ in pair to make $\mathcal{O}_{n,j}^s$ covariant. Therefore, the deformed KG equation to all orders of the effective theory is

$$-D^\mu D_\mu \phi - \frac{m^2}{\hbar^2} \phi + \sum_{j,n>D+1} C_{n,j}^s \frac{\hbar^{2q-2} m^p}{\Lambda^{n-(D+1)}} \mathcal{C} (D_{\mu_1} \cdots D_{\mu_{2q}}) \phi = 0. \quad (34)$$

Making the ansatz $\phi = \exp\left(\frac{iI}{\hbar}\right)$ and substituting it into eqn. (32) gives the deformed the Hamilton-Jacobi equation for I

$$G(X) \equiv X - m^2 + f(X) = 0, \quad (35)$$

where $X \equiv (\partial^\mu I + eA^\mu)(\partial_\mu I + eA_\mu)$ and

$$f(x) \equiv \sum_{j,n>D+1} \frac{(-1)^q C_{n,j}^s m^p x^q}{\Lambda^{n-(D+1)}}.$$

When $m/\Lambda \ll 1$ as assumed, for eqn. (35), there exists one and only one root m_{eff}^2 in $(0, 2m^2)$. In fact, since $m/\Lambda \ll 1$, $f(0)$ and $f(2m^2) \sim m^2 \frac{m}{\Lambda} \ll m^2$. Thus, one finds $G(0) < 0$ and $G(2m^2) > 0$. So there exists at one root of eqn. (35) in $(0, 2m^2)$. On the other hand, using $f'(x) \sim \frac{m}{\Lambda} \ll 1$ for $x \in (0, 2m^2)$, one finds is $G'(x) > 0$ for $x \in (0, 2m^2)$. This completes the proof of existence of one and only one root of eqn. (35) in $(0, 2m^2)$. There might be other roots which is not in $(0, 2m^2)$. However, they are not physical and discarded since they don't approach m^2 as $\Lambda \rightarrow \infty$. By solving eqn. (35), we find the classical action I satisfies

$$(\partial^\mu I + eA^\mu)(\partial_\mu I + eA_\mu) = m_{eff}^2, \quad (36)$$

where m_{eff}^2 is uniquely determined by Λ , m , and $C_{n,j}^s$.

For a fermion field, the effective operator $\mathcal{O}_{n,j}^f$ can be written as

$$\mathcal{O}_{n,j}^f = i^q \hbar^{q-1} m^p \bar{\psi} \mathcal{C} \left(D_{\mu_1}^f \cdots D_{\mu_q}^f \gamma_{\nu_1} \cdots \gamma_{\nu_{2r-q}} \right) \psi, \quad (37)$$

where integers $q, p, r \geq 0$, $p + q = n - D$, $q \geq r \geq \frac{q}{2}$, $j = \{\mathcal{C}, q, r\}$ and \mathcal{C} denotes any possible way of contracting $\mu_1 \cdots \mu_q$ and $\nu_1 \cdots \nu_{2r-q}$ in pair. Therefore, the deformed Dirac equation to all orders of the effective theory is

$$i\gamma^\mu D_\mu^f \psi - \frac{m}{\hbar} \psi + \sum_{j,n>D+1} C_{n,j}^f \frac{i^q \hbar^{q-1} m^p}{\Lambda^{n-(D+1)}} \mathcal{C} \left(D_{\mu_1}^f \cdots D_{\mu_q}^f \gamma_{\nu_1} \cdots \gamma_{\nu_{2r-q}} \right) \psi = 0. \quad (38)$$

Substituting the ansatz $\psi = \exp\left(\frac{iI}{\hbar}\right) v$ into eqn. (38), we find to the lowest order of \hbar

$$[1 - g(X)] \gamma_\mu (\partial^\mu I + eA^\mu) v = [f(X) - m] v \quad (39)$$

where $X \equiv (\partial^\mu I + eA^\mu) (\partial_\mu I + eA_\mu)$ and

$$f(x) \equiv \sum_{j,n>D+1, q \text{ is even}} \frac{C_{n,j}^f m^p x^{\frac{q}{2}}}{\Lambda^{n-(D+1)}}, \text{ and } g(x) \equiv \sum_{j,n>D+1, q \text{ is odd}} \frac{-C_{n,j}^f m^p x^{\frac{q-1}{2}}}{\Lambda^{n-(D+1)}}.$$

Using the same manipulation as before, we find that eqn. (39) reduces to

$$H(X) \equiv [1 - g(X)]^2 X - [f(X) - m]^2 = 0. \quad (40)$$

For $m/\Lambda \ll 1$, one can show that $f(0), f(2m^2) \ll m$ and $g(0), g(2m^2) \ll 1$. Thus, $H(0) < 0$ and $H(2m^2) > 0$. Moreover, $H'(x) > 0$ for $x \in (0, 2m^2)$ since $f'(x) \ll 1$ and $xg'(x) \ll 1$ for $x \in (0, 2m^2)$. Therefore, there exists one and only one root m_{eff}^2 of eqn. (40) in $(0, 2m^2)$. Eqn. (40) leads to

$$(\partial^\mu I + eA^\mu) (\partial_\mu I + eA_\mu) = m_{eff}^2, \quad (41)$$

where m_{eff}^2 is uniquely determined by Λ , m , and $C_{n,j}^f$.

IV. DISCUSSION AND CONCLUSION

The Hamilton-Jacobi method without and with the incorporation of the quantum gravity effects has been studied in this paper. Specifically, we have calculated the scalar and fermionic Hamilton-Jacobi equations for the classical action I in the background of

a $(D + 1)$ -dimensional black hole with the metric $g_{\mu\nu}$ and the electromagnetic potential A_μ . First, in Section II, we revisited the derivation of the original Hamilton-Jacobi equations for the action I of tunneling of scalar and fermionic particles from the black hole. In the framework of effective field theories constructed in Section III, the deformed model-independent KG/Dirac equations respecting covariance of spacetime and gauge invariance of A_μ have then been derived. Finally, in Section III, substituting the WKB ansatz for the scalar and fermionic wavefunctions into the deformed KG/Dirac equations, we expanded them in powers of \hbar , kept only the lowest order and hence gave the deformed Hamilton-Jacobi equations.

Our results are summarized as follows:

- (a) In the case of no quantum gravity effects, we have shown in Section II that the fermionic Hamilton-Jacobi equation for the action I can be written in the same form of the scalar one. Both can be written as

$$(\partial^\mu I + eA^\mu)(\partial_\mu I + eA_\mu) = m^2, \quad (42)$$

where A^μ is the black hole's electromagnetic potential and m is the mass of the particle.

- (b) In the case of covariant low energy quantum gravity effective models, we have shown in Section III both scalar and fermionic deformed Hamilton-Jacobi equations can be reduced to

$$(\partial^\mu I + eA^\mu)(\partial_\mu I + eA_\mu) = m_{eff}^2, \quad (43)$$

where all the quantum gravity contributions are included in only one parameter m_{eff}^2 .

As a bonus of Result (a), it provides a shortcut to calculate the action I of tunneling of a fermionic particle from the black hole. Instead of solving the complicated matrix equation (13), we can solve eqn. (42) for I . For example, such shortcut was discussed in the case of fermion tunneling from the Bardeen-Vaidya black hole[30]. Since both scalar and fermion actions satisfy the same equation (42), the Hamilton-Jacobi method relating the imaginary part of the actions to Hawking temperature indicates that Hawking temperature for scalar and fermion particles are the same. In fact, using the Hamilton-Jacobi method, Hawking temperatures were calculated for a scalar and a fermion, in the context of charged BTZ black holes[31], black strings[32, 33], *etc.*, and found the same results. As shown above, the coincidence is guaranteed by eqn. (42).

When incorporating the quantum gravity into quantum field theory, as mentioned in Introduction, there are two kinds of effective models, one of which respects covariance and the other does not. For the non-covariant effective models, the deformed Hamilton-Jacobi method with inclusion of quantum gravity effects were studied in [12–17]. There, it has been shown that the corrections to the original Hawking temperature depend on the quantum numbers of the emitted particles in a non-trivial way. In some cases, such corrections could lead to the remnant left in the evaporation[12–14]. In this paper, it is first time to investigate the deformed Hamilton-Jacobi method in covariant effective models. In this scenario, the only difference between the deformed Hamilton-Jacobi equations (43) and the original ones (42) is the parameters on RHS. As a consequence, if the original Hawking temperature of some black hole is independent of the mass of emitted particles, there will no corrections from the quantum gravity. Even if the original Hawking temperature depends on the mass m , the corrected Hawking temperature can be obtained from the original one by simply replacing m by m_{eff} . In either case above, if a black hole evaporates by the original Hawking radiations, it will also evaporate by the deformed Hawking radiations. Therefore, one may argue that covariance of spacetime in quantum gravity effective models has to be broken in order for a black hole to have the remnant left in the evaporation.

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